Problems with the error

ST552 Lecture 19

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Today

Problems with the errors

- Generalized Least Squares
- Lack of fit F-tests
- Robust regression

Generalized Least Squares

$$Y = X\beta + \epsilon$$

- We have assumed $\text{Var}(\epsilon) = \sigma^2 I$, but what if we know $\text{Var}(\epsilon) = \sigma^2\Sigma$, where σ^2 is unknown, but Σ is known. For example, we know the form of the correlation and/or non-constant variance in the response.
- The usual least squares estimates $\hat{\beta}_{LS}$ are unbiased, but they are no longer BLUE.

Let S be the matrix square root of Σ, i.e. $\Sigma = SS^T$.

Define a new regression equation by multiplying both sides by S^{-1}:

$$S^{-1}Y = S^{-1}X\beta + S^{-1}\epsilon$$
$$Y' = X'\beta + \epsilon'$$

Your Turn

Show $\text{Var}(\epsilon') = \text{Var}\left(S^{-1}\epsilon\right) = \sigma^2 I$.

Show the least squares estimates for the new regression equation reduce to:

$$\hat{\beta} = (X^T\Sigma^{-1}X)^{-1}X^T\Sigma^{-1}Y$$

- Can also show $\mathrm{Var}(\beta) = (X^T\Sigma^{-1}X)^{-1}\sigma^2$.
- The estimates: $\hat{\beta} = (X^T\Sigma^{-1}X)^{-1}X^T\Sigma^{-1}Y$ are known as **generalized least squares** estimates.
- In practice, Σ might only be know up to a few parameters that also need to be estimated.

Common cases of GLS

- Σ defines a temporal or spatial correlation structure
- Σ defines a grouping structure
- Σ is diagonal and defines a weighting structure (**Weighted Least Squares**)

In R

```r
?lm # use weights argument
library(nlme)
?gls # has weights and/or correlation argument
```

Oat yields

Data from an experiment to compare 8 varieties of oats. The growing area was heterogeneous and so was grouped into 5 blocks. Each variety was sown once within each block and the yield in grams per 16ft row was recorded.

$$\text{yield}_i = \beta_0 + \beta_1 \text{variety}_i + \epsilon_i \quad i = 1, \ldots, 40$$

$$\text{Var}\left(\epsilon_i\right) = \sigma^2, \quad \text{Cor}(\epsilon_i, \epsilon_j) = \begin{cases} \rho, & \text{block}_i = \text{if block}_j \\ 0, & \text{otherwise} \end{cases}$$

```
library(nlme)
fit_gls <- gls(yield ~ variety, data = oatvar,
  correlation = corCompSymm(form = ~ 1 | block))
```

Oat yields in R

```r
intervals(fit_gls)
```

```
## Approximate 95% confidence intervals
##
##  Coefficients:
##                  lower  est.       upper
## (Intercept) 291.542999 334.4 377.2570009
## variety2     -4.903898  42.2  89.3038984
## variety3    -18.903898  28.2  75.3038984
## variety4    -94.703898 -47.6  -0.4961016
## variety5     57.896102 105.0 152.1038984
## variety6    -50.903898  -3.8  43.3038984
## variety7    -63.103898 -16.0  31.1038984
## variety8      2.696102  49.8  96.9038984
## attr(,"label")
## [1] "Coefficients:"
##
##  Correlation structure:
##          lower      est.     upper
## Rho 0.06596382 0.3959955 0.7493731
## attr(,"label")
## [1] "Correlation structure:"
##
##  Residual standard error:
##    lower     est.    upper
## 33.39319 47.04679 66.28298
```

Lack of fit F-tests

- $\hat{\sigma}^2$ should be (if our model is specified correctly) an unbiased estimate of σ^2.
- A "model free" estimate of σ^2 is available if there are replicates (multiple observations at combinations of the explanatory values).
- If our $\hat{\sigma}^2$ from our model is much bigger than the "model-free" estimate, we have evidence of **lack of fit**.

In practice

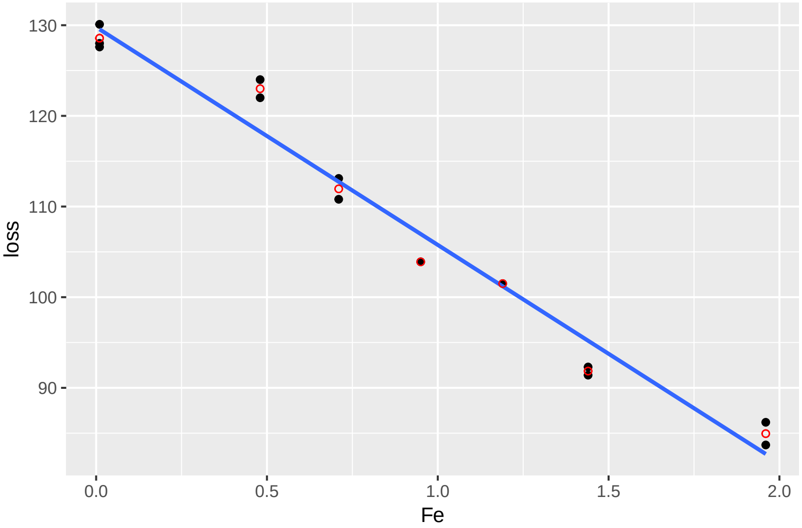
- Fit a saturated model. Compare saturated model to proposed model with an F-test. **Lack of fit F-test**.
- **Saturated**: every combination of explanatory variables is allowed its own mean (i.e. every group of replicates is allowed its own mean). A model that includes every explantory as categorical and every possible interaction between variables.

Example

```r
data(corrosion, package = "faraway")
lm_cor <- lm(loss ~ Fe, data = corrosion)
lm_sat <- lm(loss ~ factor(Fe), data = corrosion)
anova(lm_cor, lm_sat)
```

```
## Analysis of Variance Table
##
## Model 1: loss ~ Fe
## Model 2: loss ~ factor(Fe)
##   Res.Df     RSS Df Sum of Sq      F   Pr(>F)
## 1     11 102.850
## 2      6  11.782  5    91.069 9.2756 0.008623 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```r
# significant lack of fit
```



130
120
110
100
90
loss
0.0
0.5
1.0
1.5
2.0
Fe

Robust regression

Remember to define our least squares estimates we looked for β to minimise

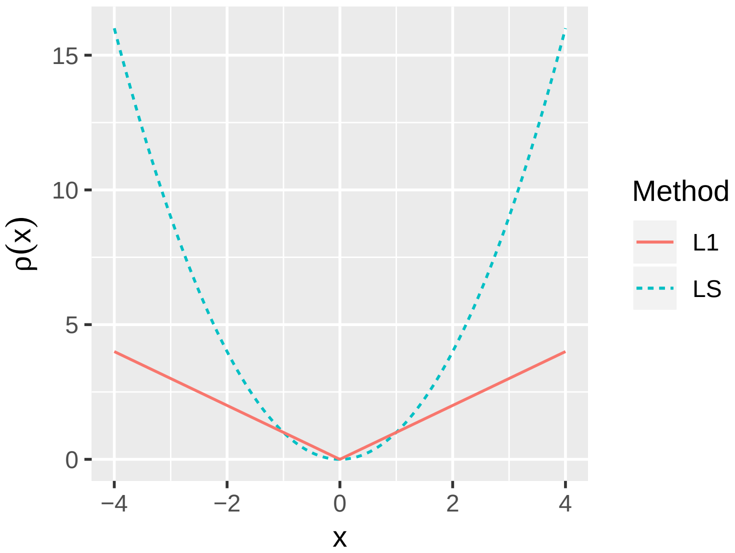
$$\sum_{i=1}^{n} \left(y_i - x_i^T \beta\right)^2$$

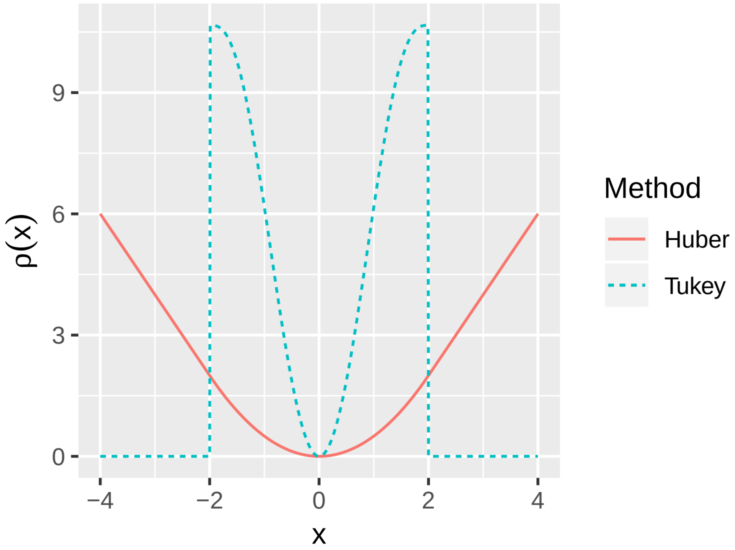
In practice, since we are squaring residuals, observations with large residuals carry a lot of weight. For, robust regression, we want to downweight the observations with large residuals.

The idea of M-estimators is to extend this to the general situation where we want to find β to minimise

$$\sum_{i=1}^{n} \rho(y_i - x_i^T \beta)$$

where $\rho()$ is some function we specify.


15
10
5
0
ρ(x)
−4
−2
0
2
4
x
Method
L1
LS



9
6
3
0
ρ(x)
−4
−2
0
2
4
x
Method
Huber
Tukey

$$\sum_{i=1}^{n} \rho(y_i - x_i^T \beta)$$

- Least squares: $\rho(e_i) = e_i^2$
- Least absolute deviation, L_1 regression: $\rho(e_i) = |e_i|$
- Huber's method

$$\rho(e_i) = \begin{cases} e_i^2/2 & \text{if } |e_i| \leq c \\ c|e_i| - c^2/2 & \text{otherwise} \end{cases}$$

- Tukey's bisquare

$$\rho(e_i) = \begin{cases} \frac{1}{6}(c^6 - (c^2 - e_i^2)^3) & |e_i| \leq c \\ 0 & \text{otherwise} \end{cases}$$

The models are usually fit in an iterative process.

Least trimmed squares

Minimise the smallest residuals

$$\sum_{i=1}^{q} e_{(i)}^2$$

where q is some number smaller than n and $e_{(i)}$ is the ith smallest residual.

One choice, $q = \lfloor n/2 \rfloor + \lfloor (p+1)/2 \rfloor$

Annual numbers of telephone calls in Belgium