Non-normal inference like the bootstrap CI / hypothesis tests

Randomization/Permutation tests

CI's

ST552 Lecture 15

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Randomization Test

The F-tests (and t-tests) rely on the Normal error assumption.

In a randomized experiment, the randomization provides a basis for inference (no i.i.d sampling from populations required) and results in **randomization tests**.

The same procedure can be used in observational studies, with the assumption that nature ran the experiment for you, i.e. it's like units were assigned to values of the explanatory variables at random.

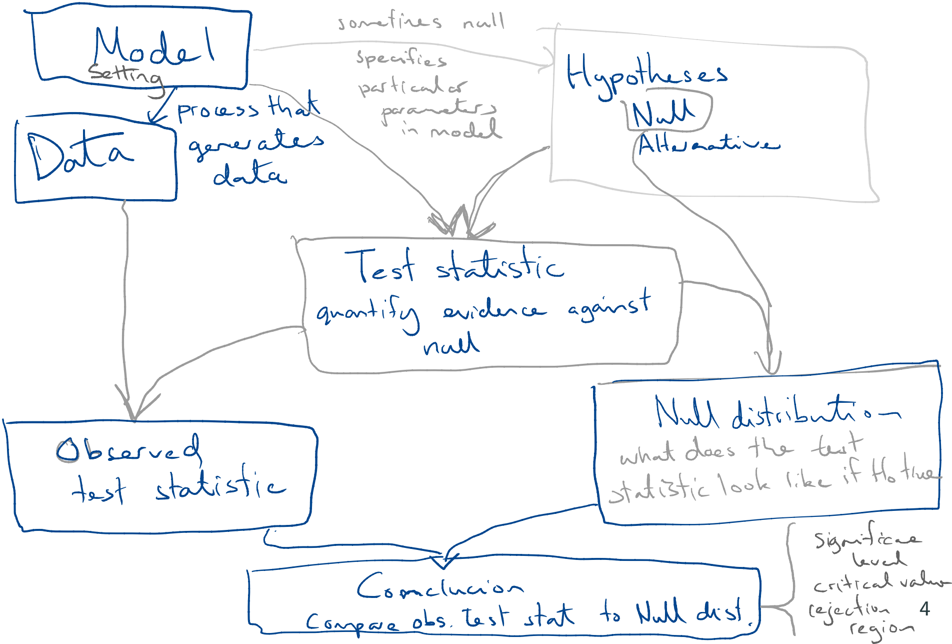
Some people call randomization tests used for observational data, **permutation tests**.

↑ but it's not
the only this word is used

Your Turn

What are the key ingredients in an hypothesis test?

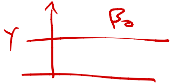
Hypothesis Testing



Randomization Test

Usual F-test

Model:
$y = X\beta + \varepsilon \quad \varepsilon \sim N(0, \sigma^2 I)$



Analog:
(Overall F-test example)

Model: Randomized experiment

Null: Treatments have no effect on response

Test statistic: *(Up to us)* Let's use overall regression F-statistic.

Null distribution: Randomization distribution of the test statistic.

the dist. of test statistic when null is true
under all possible randomizations of units to treatments

The randomized experiment model

n experimental units

$$\boxed{u_1 \quad u_2 \quad \dots \quad u_n}$$

assign units to rows. of design matrix at random

apply treatments

observe responses

Fixed

$$X = \begin{pmatrix} 1 & x_{11} & \dots & x_{1(p-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & \dots & x_{n(p-1)} \end{pmatrix}$$

↑ column describe treatments

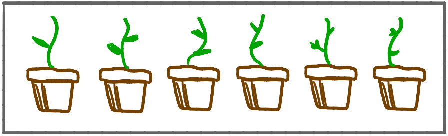
$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

y_i the observed response for the the unit that was randomly assigned to the ith row of the design matrix.

Example: Growing tomatoes

$\cancel{n}$ experimental units

6



3 2 6 1 5 4

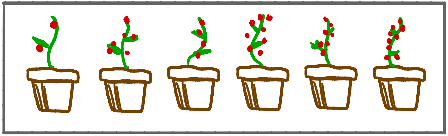
at random

$$X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

β_0 fertilizer water

apply treatments

wait ..... count tomatoes



$$y = \begin{pmatrix} 8 \\ 6 \\ 2 \\ 9 \\ 7 \\ 4 \end{pmatrix}$$

Null distribution

If the null is true, treatments have no effect on response.

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

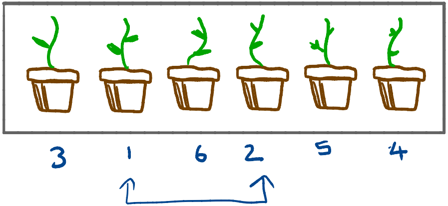
y_i the observed response for the the unit that was randomly assigned to the ith row of the design matrix.

If the null is true, I see the same set of y_i, just in different order based on the output of my randomizing units to treatment.

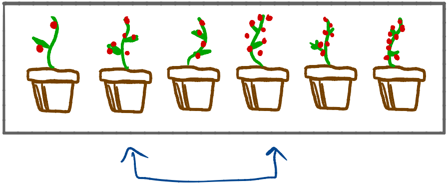
Null distribution: the distribution of the test-statistic for all permutations of y_i

An equally likely output of the tomato growing experiment

n experimental units



$$X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$



$$y = \begin{pmatrix} 6 \\ 8 \\ 2 \\ 9 \\ 7 \\ 4 \end{pmatrix}$$

The null distribution

Observed data gives overall F-statistic: 0.132

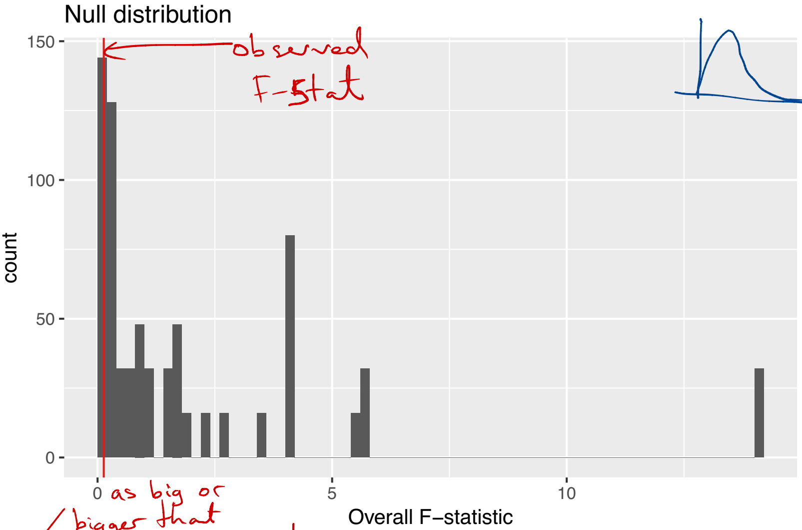
Equally likely outcomes under the null hypothesis:

$$\begin{pmatrix} 8 \\ 6 \\ 2 \\ 9 \\ 7 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 \\ 8 \\ 2 \\ 9 \\ 7 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 8 \\ 9 \\ 7 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 \\ 2 \\ 8 \\ 9 \\ 7 \\ 4 \end{pmatrix}, \quad +716 \text{ other possibilities}$$

Equally likely F-statistics under the null hypothesis:

0.132, 0.244, 5.534, 0.244 + 716 F-stats

The null distribution



as big or bigger that we observed

$624/\underline{720} = 0.87$ ← p-value

Faraway: Galapagos

In lab:

```r
library(faraway)
lmod_small <- lm(Species ~ Nearest + Scruz,
  data = gala)
lms <- summary(lmod_small)
obs_fstat <- lms$fstat[1]

nperms <- 4000
fstats <- numeric(nperms)
for (i in 1:nperms){
  lmods <- lm(sample(Species) ~ Nearest + Scruz,
    data = gala)
  fstats[i] <- summary(lmods)$fstat[1]
}
```

In practice

- Easiest to justify when you actually have a randomized experiment
- The choice of test statistic can be important for useful performance and interpretation:
  For example, if the treatments affect the variance of the response, not the means, using the overall F-stat may fail to reject the null (treatment has no effect) with high probability even when sample sizes are large.
  Sometimes it's reasonable to add an assumption on the alternative, i.e. treatments have an additive effect.

→ linear on mean

↓ assume a regression model
could have interaction