Some details to tidy up

ST552 Lecture 7

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Summary of last week

For the linear regression model

$$Y = X\beta + \epsilon$$

where $\mathrm{E}(\epsilon) = 0_{n\times 1}$, $\mathrm{Var}(\epsilon) = \sigma^2 I_n$, and the matrix $X_{n\times p}$ is fixed with rank p.

The least squares estimates are

$$\hat{\beta} = (X^T X)^{-1} X^T Y$$

Furthermore, the least squares estimates are BLUE, and

$$\mathrm{E}\left(\hat{\beta}\right) = \beta, \qquad \mathrm{Var}\left(\hat{\beta}\right) = \sigma^2 (X^T X)^{-1}$$

We have not used any Normality assumptions to show these properties.

Today

- Verify:

$$\mathsf{E}\left(\hat{\sigma}^2\right) = \mathsf{E}\left(\frac{1}{n-p}\sum_{i=1}^{n} e_i^2\right) = \sigma^2$$

- Add Normal assumption to get inference on regression coefficents.

Go over the estimation of σ

Strategy: Write e_i^2 as a linear combination of uncorrelated variables, ϵ_i.

Write correlated residuals as combination of uncorrelated errors

Claim:

$$||e||^2 = \epsilon^T (I - H)\epsilon$$

Your turn at home:

1. Show $(I - H)\epsilon = e$. Hint: substitute $\epsilon = Y - X\beta$, expand and use properties of H.
2. Show $||e||^2 = e^T e = \epsilon^T (I - H)\epsilon$. Hint: substitute in $e = (I - H)\epsilon$ from above and use properties of $(I - H)$.

Find expected value of $||e||^2$ in terms of trace$(I - H)$

Show $\mathrm{E}\left(\epsilon^T(I-H)\epsilon\right) = \sigma^2\text{trace}(I-H)$

Hint

$$x^T A x = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j A_{ij}$$

where

$$x = (x_1, x_2, \ldots, x_n)^T, \quad A = \begin{pmatrix} A_{11} & A_{12} & \ldots \\ A_{21} & A_{22} & \ldots \\ \vdots & & \end{pmatrix}_{n\times n}$$

Find expected value of $||e||^2$ in terms of trace$(I - H)$

$$\mathrm{E}\left(\epsilon^T (I - H)\epsilon\right) =$$

Find trace$(I - H)$

Show

$$\text{trace}(I - H) = n - p$$

Hint:

$$\text{trace}(A + B) = \text{trace}(A) + \text{trace}(B)$$

$$\text{trace}(AB) = \text{trace}(BA)$$

$$\text{trace}(I - H) =$$

Put it all together

$$\mathrm{E}\left(\hat{\sigma}^2\right) =$$

Inference on the regression coefficients

Normality assumption

Assume $\epsilon \sim N(0, \sigma^2 I)$.

Important reminders:

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Leads to:

$$Y \sim N(\quad , \quad)$$

$$\hat{\beta} \sim N(\quad , \quad)$$

Inference on individual parameters

With the addition of the Normal assumption, it can be shown that

$$\frac{\hat{\beta}_j - \beta_j}{SE(\hat{\beta}_j)} \sim t_{n-p}$$

leads to the usual construction of tests and confidence intervals for single parameters.

Exercises

See handout.