ST552 Midterm

Winter 2016

Answer the questions in the spaces provided on this exam.

Name: SOLUTIONS

- You have 50 minutes to complete the exam.
- There are 3 questions. Answer all of the questions.
- Please
  - do not look at the exam until I tell you and
  - stop writing when I announce that the exam is over.
- There is one page of statistical tables at the end of the exam. You may remove the page of tables if you desire.

Question	Points	Score
1	15	
2	8	
3	17	
Total:	40	

1. (a) Derive the variance-covariance matrix of the least squares estimates of $\boldsymbol{\beta}$. You should begin by stating the multiple linear regression model in matrix form, along with any assumptions you require. (10)

$Y = X\beta + \varepsilon$ form 1

$Y_{n\times 1}$ response vector
$X_{n\times p}$ matrix of covariates
$\beta_{p\times 1}$ vector of parameters
$\varepsilon_{n\times 1}$ vector of random errors

dimensions 1

Assumptions: X fixed and $\text{rank}(X) = p$ 1
$E(\varepsilon) = 0 \quad Var(\varepsilon) = \sigma^2 I$ 2

Least squares estimates $\hat{\beta} = (X^T X)^{-1} X^T y$

$$Var(\hat{\beta}) = Var\left((X^T X)^{-1} X^T y\right)$$

$$= (X^T X)^{-1} X^T \, Var(y) \left((X^T X)^{-1} X^T\right)^T$$

$Var(aU) = a \, Var \, U a^T$

$$= (X^T X)^{-1} X^T \, Var(y) X (X^T X)^{-1}$$

$(AB)^T = B^T A^T$
$(A^{-1})^T = (A^T)^{-1}$

4
correct steps

$$= (X^T X)^{-1} X^T \sigma^2 I \, X (X^T X)^{-1}$$

$Var(y) = Var(\varepsilon) = \sigma^2 I$

1
reasoning

$$= \sigma^2 (X^T X)^{-1} \cancel{X^T X} (X^T X)^{-1}$$

$$= \sigma^2 (X^T X)^{-1}$$

(b) Imagine the errors have non-zero mean. For example, that $\mathrm{E}(\epsilon) = c$, where c is a constant $n \times 1$ vector. Are the least squares estimates still unbiased? Justify your answer. (5)

$$
\begin{aligned}
E(\hat{\beta}) &= E\left((X^T X)^{-1} X^T y\right) \\
&= (X^T X)^{-1} X^T E[X\beta + \varepsilon] \\
&= (X^T X)^{-1} X^T X \beta + (X^T X)^{-1} X^T c \\
&= \beta + (X^T X)^{-1} X^T c
\end{aligned}
$$

3

No. (unless $c \perp$ to X)

1 1

2. An observational study was conducted to explore the relationship between the brain size of mammals and their gestation length and lifetime, after accounting for body size.

   The following models were fit to a dataset that contains data on ~~62~~ 55 mammals:

   $$\log(\text{Brain size})_i = \beta_0 + \beta_1 \log(\text{Body Size})_i + \beta_2 \log(\text{Gestation Length})_i + \beta_3 \log(\text{Lifetime})_i + \epsilon_i$$
   $$\log(\text{Brain size})_i = \beta_0 + \beta_1 \log(\text{Body Size})_i + \epsilon_i$$

   The estimates of σ^2 are 0.31 ($\hat{\sigma}^2_\Omega$) and 0.~~48~~ .51 ($\hat{\sigma}^2_\omega$) for the two models respectively.

   (a) Conduct an F-test to compare the two models. (6)

$RSS_\Omega = 0.31(55-4) = 15.81$ 1

$RSS_\omega = 0.51(55-2) = 27.03$

$$F = \frac{(RSS_\omega - RSS_\Omega)/(p-q)}{\underbrace{RSS_\Omega / n-p}_{\hat{\sigma}^2_\Omega}}$$

$$= \frac{(27.03 - 15.81)/2}{0.31}$$

$$= \frac{5.61}{0.31} = 18.09$$ 2

Compare to $F_{2,\cancel{58}\,51}(0.95) \approx 3.15 \ll F$ 1

Reject null. 1

There is convincing evidence at least one of Gestation length and Lifetime is associated with mean brain size, after accounting for body size.

(b) An 95% confidence region is estimated for (β_2, β_3), in the larger model. Would you expect the point $(0, 0)$ to be inside or outside of the confidence region? Justify your answer. (2)

1

Outside, since the null, $H_0: (\beta_2, \beta_3) = (0, 0)$ was rejected at the 5% level, it would not be contained in the 95% confidence region.

1 reasoning

3. An experiment was conducted to explore the relationship between the *lifetime* (measured in days) and sexual activity of fruitflys.

   125 fruit flys were divided randomly into 5 treatment groups, each of 25 flys. Each treatment was designed to simulate a different level of sexual activity, with levels: *none*, *one*, *many*, *low* and *high*.

   The *thorax length* of each male was also measured as this was known to affect lifetime.

   One observation in the *many* group was lost.

   The following model was fit:

$$\text{Lifetime}_i = \beta_0 + \beta_1 \text{Thorax Length}_i + \beta_2 \text{one}_i + \beta_3 \text{low}_i + \beta_4 \text{many}_i + \beta_5 \text{high}_i + \epsilon_i$$

   where *one*, *low*, *many*, and *high* are indicator variables for the respective treatment groups.

   Results from the least squares fit are given below.

$$\hat{\beta} = \begin{pmatrix} -48.8 \\ 134.3 \\ 2.6 \\ -7.0 \\ 4.1 \\ -20.0 \end{pmatrix}, \qquad (X^TX)^{-1} = \begin{pmatrix} 1.06 & -1.22 & -0.05 & -0.04 & -0.07 & -0.08 \\ -1.22 & 1.46 & 0.02 & -0.00 & 0.03 & 0.05 \\ -0.05 & 0.02 & 0.08 & 0.04 & 0.04 & 0.04 \\ -0.04 & -0.00 & 0.04 & 0.08 & 0.04 & 0.04 \\ -0.07 & 0.03 & 0.04 & 0.04 & 0.08 & 0.04 \\ -0.08 & 0.05 & 0.04 & 0.04 & 0.04 & 0.08 \end{pmatrix}$$

$$\hat{\sigma} = 10.54,$$

   (a) Conduct a t-test of the null hypothesis that $\beta_1 = 0$. (6)

$t = \dfrac{134.3}{10.54\sqrt{1.46}} \doteq 10.5$

(1) (1)

compare to $t_{124-6}(0.975) = \underbrace{t_{118}(0.975)}_{(1)} \approx 1.98$

reject null. (1)

There is strong (1) evidence that thorax length is associated with mean lifetime, after accounting for the treatment effects. (1)

(b) For two flys with the same thorax length, show the difference between the expected lifetime for a fly in the *one* treatment group and the expected lifetime for a fly in the *low* treatment group is $\beta_2 - \beta_3$. (3)

① ②

①: $E(y_i \mid one_i = 1, \text{thorax}, \text{all others} = 0) = \beta_0 + \beta_1 \text{thorax} + \beta_2$ ①

②: $E(y_i \mid low_i = 1 \text{ " " " "}) = \beta_0 + \beta_1 \text{thorax} + \beta_3$ ①

① − ② $= (\beta_0 + \beta_1 \text{thorax} + \beta_2) - (\beta_0 + \beta_1 \text{thorax} + \beta_3)$

$= \beta_2 - \beta_3$ ①

(c) Construct a 95% confidence interval for $\beta_2 - \beta_3$. (6)

$\hat\beta_2 - \hat\beta_3 = 2.6 - -7.0 = 9.6$ ②

$SE(\hat\beta_2 - \hat\beta_3) = 10.54 \times \sqrt{Var\, \hat\beta_2 + Var\, \hat\beta_3 - 2Cov(\beta_2, \beta_3)}$

$= 10.54 \times \sqrt{0.08 + 0.08 - 2(0.04)}$

$= 10.55 \times \sqrt{0.08}$

$= 2.98$ ③

$9.6 \pm 1.98(2.98)$ ①

$= (3.7, 15.5)$

(d) Interpret your interval from (c) in the context of the study. (2)

With 95% confidence, for flys with the same thorax length, flys receiving the one treatment live between 3.7 and 15.5 days longer than those receiving the low treatment.

(randomized experiment:

... the low treatment reduces lifetime by between 3.7 & 15.5 days compared to the one treatment)

End of exam.