# Multiple Linear Regression

ST552 Lecture 4

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# Today

- Finish last time's slides
- Matrix warmup
- Multiple Linear Regression
- Matrix setup
- Least squares estimates

# Matrix warmup

**See handout**

# Multiple Linear regression

We have $n$ observations of a response and a set of explanatory variables, $(y_i, x_{i1}, x_{i2}, \ldots, x_{i(p-1)}), i = 1, \ldots, n$ where the $y_i$ are generated according to the model,

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_{p-1} x_{i(p-1)} + \epsilon_i$$

where $\epsilon_i$ are independent and identically distributed with expected value 0, and variance $\sigma^2$.

(Note: intercept and $p$)

# Matrix form

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & x_{11} & x_{12} & \dots & x_{1(p-1)} \\ 1 & x_{21} & x_{22} & \dots & x_{2(p-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n1} & x_{n2} & \dots & x_{n(p-1)} \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{p-1} \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{pmatrix}$$

$$y = X\beta + \epsilon$$

where

$$y_{n\times 1} = (y_1, y_2, \dots, y_n)^T$$

$$\epsilon_{n\times 1} = (\epsilon_1, \epsilon_2, \dots, \epsilon_n)^T$$

$$\beta_{p\times 1} = (\beta_0, \beta_1, \dots, \beta_{p-1})^T$$

$$X_{n\times p} = \begin{pmatrix} 1 & x_{11} & x_{12} & \dots & x_{1(p-1)} \\ 1 & x_{21} & x_{22} & \dots & x_{2(p-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n1} & x_{n2} & \dots & x_{n(p-1)} \end{pmatrix}$$

# Your Turn

Write out the design matrix, $X$, for the following models, using the data given below:

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i$$
$$y_i = \beta_1 x_{i1} + \beta_2 x_{i1}^2 + \epsilon_i$$
$$y_i = \beta_0 + \beta_1 1_{\{x_{i1}>0\}} + \epsilon_i$$

where $1_{\{.\}}$ is an indicator variable that takes the value 1, when the condition in the argument is true, and 0 otherwise.

$$\mathbf{x}_1 = (x_{11}, x_{21}, \ldots, x_{n1}) = (-2, -1, 0, 1, 2)$$
$$\mathbf{x}_2 = (x_{12}, x_{22}, \ldots, x_{n2}) = (1, -1, 1, 1, -1)$$

# Fitted values and residuals

$$\hat{y} = (\hat{y_1}, \ldots, \hat{y_n})^T = X\hat{\beta}$$
$$e = \hat{\epsilon} = (e_1, \ldots, e_n)^T = Y - X\hat{\beta}$$

- How will we estimate $\beta$?
- What properties do the estimates have?

# Least squares derivation (sketch)

Least squares estimates are:

$$\hat{\beta} = (X^T X)^{-1} X^T y$$

Derive the matrix form for the residuals and fitted values:

$$\hat{y} = X\hat{\beta}$$
$$e = y - X\hat{\beta}$$

$$H =$$

# Your Turn

Show the following properties of $H$:

- $H$ is symmetric, so is $(I - H)$
- $H$ is idempotent ($H^2 = H$), and so is $I - H$
- $X$ is invariant under $H$ (i.e. $HX = X$)
- $e \perp X$ (i.e. $X^T e = 0$, the residuals are orthogonal to the columns of $X$)
- $(I - H)X = 0$
- $(I - H)H = H(I - H) = 0$

# Your turn

For scalar random variable $X$ and constant scalars $a$ and $b$, complete the following:

$$\mathsf{E}\left(aX + b\right) =$$
$$\mathsf{Var}\left(aX + b\right) =$$

# Random vectors